Chapter 1 - Can Mathematicians be Poets?

I vividly remember in high school reading the following poem in the last page of my geometry textbook,

There once was a lady named Bright
Who could travel far faster than light,
She set out one day,
In a relative way,
And came back the previous night.

Oddly enough, the poem had no title and there was no mention of the author. Today, I still do not know who wrote it but in spite of that I find it beautiful. Although at the time I could not comprehend its full meaning, it definitely made me think. I tried to imagine how such an incredible tale could be possible in the real world. But how can we define reality? There is no doubt that what we experience through our senses is only part of it. For example, you are presently reading this book but at the same time many different biological processes are taking place within your body, not to mention what is going on through your mind. Some of these processes might be conscious, such as the fact that you are hungry or thirsty, while others are unconscious, like what you will remember after finishing this page. And the same is true with respect to the exterior world. When you walk out of your home and stop somewhere, you really have no idea of what exists directly below your feet or far above your head. As such, I believe that our perception of reality is dependent on the knowledge that we possess both of ourselves and what surrounds us.

Several decades have passed since I first read this poem. During this period I have acquired both knowledge and experience, so do I understand it better? We will discuss this at the end of Chapter 2. But even if I comprehend its physical implications, can I know the original intention of the poet? The inevitable answer is no. One simple reason is that even the poet could not know it, which in this particular case is very unlikely. All the same, every time a poet writes, he or she is codifying in some way what they feel and think. The code most often used is the letters of the alphabet. Their adequate mixture in smaller or greater number gives rise to words. In turn, the arrangement of words creates a sentence which may or may not have meaning. Given that the English language consists of many thousands of different words, the number of possible combinations is practically infinite, considering both long and short sentences. It is therefore more of an art than a science to be able to express oneself correctly and clearly. For that reason, poets are and will always be considered as literary artists.

Nonetheless, something is always lost when transforming emotions into words. And the problem is not only due to the limitations of language itself. To some degree, both doubt and confusion coexist in the heart and mind of any artist. Great poets are those who can reduce these obstacles to a minimum. As a result, their works codify almost to perfection the subject at hand. To put it another way, such poets are able to use language so that little or nothing seems to be missing between what they initially felt and thought and what they later wrote. Masterpieces share this common feature and brilliantly encompass other topics such as mythology and history.

The history of symbols in mathematics

Most people believe that mathematics is just about numbers and geometric figures. That is far from true. Nowadays there is virtually no field of human knowledge which does not include some form of mathematics. Why is this so? While physicists probe natural laws, biologists study living organisms; psychologists examine human behavior; mathematicians think. In fact, they contemplate abstract notions such as space, quantity, structure and change. On the other hand, physics deals with the concept of space; biology with that of structure and psychology with deviation or change from the norm. Mathematics is one of several possible codes for human thought, and music is another example.

If we assume that numbers and letters are symbols, then mathematics consists of all possible combinations between different types of symbols and signs. An example of a sign is the common operation of addition in elementary arithmetic. Just like a poet, a mathematician can create a new symbol or sign. Given that numbers are infinite, then mathematics is bound only by the limits of human imagination. In general, mathematical problems can have anything between one and an infinite number of solutions. Similarly, there may exist one or more ways to arrive at each result. Hence and in parallel to poetry, mathematics can be regarded as the art of thought.

In order to better comprehend mathematics, it is essential to learn about the history of symbols. Presently, our number system is based on the familiar symbols: 1, 2, 3, 4, 5, 6, 7, 8, and 9. Yet, these were not always written as such. Actually they are the “symbolic descendants” of the Devanagari numerals that were used in India sometime during the first centuries AD, as shown below:

1	2	3	4	5	6	7	8	9
□	□	□	□	□	□	□	□	□

While some numerals resemble our present-day numbers, others are significantly different. So, the way numbers were written evolved over the millennia. Zero is missing because it is a very special number. Its origin has been traced to Indian texts and also to Chinese manuscripts belonging to a later period. All the same, the creation of a symbol for zero gave rise to our current decimal number system. Its most striking feature is that it is a positional number system based on zero. In other words, the first positive even integer is zero; after nine we have ten which consists of a one and a zero; following nineteen is twenty which consists of a two and a zero, and so on. This simple property provided humanity with the necessary tool to express any quantity in an efficient and concise manner. As far back as the ninth century BC, the Etruscans used particular numerals that centuries later evolved into the well-known Roman numerals. But if I were to use these to express my year of birth, 1966, it would be MXMLXVI, thus requiring seven symbols instead of only four. Hence, any sort of calculation such as addition and multiplication is also done more easily using the positional number system.

Another consequence of the positional number system is the number line where numbers to the left of zero are negative and those to the right of zero are positive, as shown below:

◄ … -5 -4 -3 -2 -1 0 1 2 3 4 5 … ►

Zero, and all the positive and negative numbers are called integers. Natural numbers are only the positive integers excluding zero. Rational numbers are all those that can be expressed as the ratio of any two integers, such as ½ or ⅝. Irrational numbers are those that cannot be represented as a fraction and consequently are non-terminating in their decimal representation, such as e = 2.71828… and π = 3.14159…. Thus real numbers include the integers and both rational and irrational numbers. As a result, the number line is of fundamental importance to many areas of mathematics, such as geometry and calculus. For example, in geometry the concept of distance between two points corresponds to the difference between the respective integers on the number line. Likewise, a surface involves two number lines that are perpendicular to each other and a geometric solid requires three number lines that are mutually perpendicular. Calculus makes use of all this along with the real numbers to calculate any length, area or volume.

Returning to our history of symbols, Arab scholars in the ninth century AD developed algebra and were responsible for introducing the decimal number system in Europe using Hindu-Arabic numerals. Such influence was crucial for the evolution of mathematics in the rest of the world. It should be emphasized that the diffusion of mathematical knowledge was indeed very slow, and long periods when no significant advances occurred were quite frequent. As such the translation of the Hindu numerals into Latin only took place in the 12th century, and their widespread use required three more centuries. The mathematical signs "+" for addition and "–" for subtraction are due to Nicola d'Oresme (1323-1382). The sign "=" for equals only appeared in 1557 and is credited to Robert Recorde (1510-1558). The sign "×" for multiplication was created by William Oughtred (1574-1660). Later, Johann Rahn (1622-1676) invented the sign "÷" for division. In 1655 John Wallis (1616-1703) devised the symbol "∞" for infinity used in calculus and twenty years later Gottfried Wilhelm Leibniz (1646-1716) conceived the symbol "∫" for integration. In the 18th century, Leonhard Euler (1707-1783) invented the symbol "∑" for summation, whereas Carl Friedrich Gauss (1777-1855) concocted the symbol "∏" for product in the following century. These are only some of the many signs and symbols existing in mathematics today.

Our friends, the primes!

The previous idea that both numbers and letters are symbols is not new. As a matter of fact, in 1931 a logician named Kurt Gödel (1906-1978) extended such a view to include mathematical signs. In other words, Gödel considered numbers, letters and signs as symbols. He presented this to the academic world in what are presently known as Gödel's incompleteness theorems. These theorems demonstrate that mathematics has certain intrinsic limitations which mathematicians were not aware of. As one can imagine, the impact of this breakthrough was profound. But what motivated Gödel into this line of reasoning? Many have heard about the "liar paradox" which states "This statement is untrue," but that means that if it is true then it is untrue and if it is untrue then it must be true. There is no doubt that this is a paradox. Nonetheless, something can be learned, that is, the notion of truth cannot be defined within the English language. If the reader finds this amazing, then wait and see how Gödel applied this logic to mathematics or more precisely to arithmetic. In very basic terms, his two incompleteness theorems showed that any mathematical theory based on arithmetic cannot be both consistent and complete; and that there are statements in arithmetic that are true, but cannot be derived from the rules of arithmetic. Remember that we had previously seen that a mathematical problem could have anywhere between one and an infinite number of solutions, yet this is not always correct. There is still the case of no solution at all, or simply that the problem is unsolvable. If truth be told, this is exactly what Gödel concluded: that there are mathematical problems that can never be solved!

Just stop and think about its implications. Imagine you are a mathematician and you had to select a problem to work on, how could you know that the problem chosen is not one of those having no solution? The answer is that we cannot know beforehand whether a problem is unsolvable. Instead, suppose that some other problem is shown to have no solution; will it remain as such in the next 100 years? Yes! Once a mathematical problem has been proven to be unsolvable, it will remain unsolvable for all eternity independent of what mathematicians may discover within this period and beyond.

Therefore, how was Gödel able to prove this? To put it metaphorically, his goal was to establish a playing field in which all players of mathematics would be on an equal footing. The long sought strategy consisted of two simple steps. First, to assign a natural number to every existing symbol. In doing so he created a scheme by which any mathematical statement could be transformed into a unique calculation. Second, at this point, the reader must realize that everything is reduced to numbers and consequently the rules of arithmetic can be applied. But, after all, what are operations such as addition and division? They connect numbers to numbers in different ways, that is, relations are created among letters, numbers and signs. As a result, the game is over.

Because natural numbers include all the positive integers excluding zero, what did Gödel use for his grand design? Surprisingly or not, the topic of this book is the answer, i.e., the prime numbers! From here on our attention will focus on a special class of integers that have the distinct property of being divisible only by themselves and one. Such numbers are called primes or prime numbers. The first ten primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. Can you guess the next prime? **Q1** (The answers are at the end of the book). The Greek mathematician Euclid formulated the fundamental theorem of arithmetic stating that every positive integer can be written as a product of a finite number of primes. For this reason, the prime numbers stand out as the building blocks of all arithmetic.

Now that we know that the primes are the secret behind his encoding, let us present some of the definitions behind the Gödel numbering system:

First definition: The number one is assigned to the word "*not*"; the number two is assigned to the word "*or*"; three is assigned to the words "*If...then*"; four is assigned to the words "*There is an*"; five is assigned to the sign "="; six is assigned to the number zero; seven is assigned to the words "*Immediate successor of*"; eight is assigned to the punctuation mark "("; nine is assigned to the punctuation mark ")"; and ten is assigned to the punctuation mark "," or comma.

Second definition: Variables such as the commonly used x, y and z are assigned distinct prime numbers greater than ten.

Third definition: Every mathematical statement is a finite sequence of symbols.

We follow with an example. The statement: "*If* $x = y$ *then* $y = z$" in terms of Gödel numbering becomes $2^3 \times 3^{11} \times 5^5 \times 7^{13} \times 11^3 \times 13^{13} \times 17^5 \times 19^{17}$ or an astronomical number. Here x, y and z are assigned the Gödel numbers of 11, 13 and 17, respectively. Hence, the general form is always the sequence of the primes raised to the corresponding Gödel number. Although the association of a symbol with a Gödel number is arbitrary, the resulting numerical value is unique for every mathematical statement. Such a system sets the ground for the two incompleteness theorems formulated by Gödel.

Types of primes and related problems

Around 300 BC Euclid proved that the prime numbers are infinite in number. How did he achieve this? Basically, by pondering the meaning of infinity and its implications. When we say that primes are infinite in number what we are actually stating is that primes continue forever. This means that there cannot be a number that is the largest prime because this would represent the last prime thus causing a contradiction. Being aware of this Euclid assumed that the number of primes is finite. For example, consider 2, 3, 5, and 7. Multiply all of them and add one to the final result such that $2 \times 3 \times 5 \times 7 + 1 = 211$. According to the previously mentioned fundamental theorem of arithmetic every positive integer is a product of a finite number of primes. Let's apply this to 211. As the reader can easily verify 211 is not divisible by any of the above primes hence it must be itself a prime number. But this is absurd! Thus the only logical explanation is that our initial assumption is wrong, that is, primes are infinite in number. Would this still work if instead of adding one we subtracted one? **Q2**

The world of mathematics is full of enigmas and among them there are those that can be easily understood by a high-school student, yet their solutions have eluded the most brilliant minds for centuries. One such outstanding mathematical problem is the famous Goldbach conjecture, named after the Prussian mathematician and historian Christian Goldbach (1690-1764). It states that it is possible to write every even number greater than 2 as the sum of two primes, such as $4 = 2 + 2$ and $8 = 3 + 5$, and so on. Can you find (as a counterexample) an even number that cannot be represented as the sum of two primes? **Q3**. Almost all mathematicians believe that this conjecture is true, but most annoyingly, nobody has ever been able to prove it! In fact, this problem could be one of those according to Gödel's incompleteness theorems that is unsolvable. Nevertheless, we will examine it further in Chapter 5.

Marin Mersenne (1588-1648) was a French monk who conjectured that all the numbers of the form $2^n - 1$ where n is a positive integer are primes. For its time, this was a bold statement believed to be ultimately correct! Nowadays we know that n must be prime (this is a necessary though not a sufficient condition) and so, those numbers of the form $2^p - 1$ resulting in primes are presently called Mersenne primes. At the time of this writing, forty-seven Mersenne primes have been found. The first of these is $2^2 - 1 = 3$ and the last one is the largest known prime, $2^{43112609} - 1$ which is 12,978,189 decimal digits long. Please note that 2 and 43112609 are both prime numbers as expected. To date, nobody knows whether Mersenne primes are infinite in number. We will talk more about them in Chapter 5.

Another mystery is the twin prime conjecture. We call twin primes a pair of primes that differ by two, such as (5, 7) or (11, 13). It has been conjectured, but once again never proven, that there are infinitely many twin primes. We will revisit these intriguing primes in Chapter 8.

Why are these purely theoretical problems so important? As often happens in science, the methods and techniques developed to solve mathematical questions have also proven fruitful in other disciplines, such as physics and chemistry, and at times have paved the way to entirely new and exciting problems. In addition, many mathematical results have had an unexpected impact in the high-tech industry. For example, the famous Pentium bug (an error in the division operation of the Pentium processor), which was not detected during the production process, resulted in the loss of millions of dollars to Intel Corporation for having to replace all those faulty chips. Curiously enough, the problem was found accidentally by Thomas Nicely, a professor of mathematics, when attempting to calculate a mathematical constant directly related to the twin primes.

Another interesting conjecture is due to the German mathematician, Lothar Collatz (1910-1990), who formulated it in 1937, and is presently known as the $3n + 1$ problem. We can define it as follows: take any positive integer; if it is odd, multiply it by three and add one; if the number you chose is even, divide it by two. Now, we can go on and repeat once again this procedure to the number that we got, and again, and again… The Collatz problem states that for every positive integer, this procedure (called "iteration") must always terminate in 1. For example, if we start with 3, which is an odd integer, multiply it by 3 and add 1, we will get 10. Ten is even, and divided by 2 will give us 5, which is odd, and multiplied by 3 plus 1, is 16, which is even, and divided by 2 will give 8, and then 4, and then 2, and finally we will end with 1. Now, try for yourself the same procedure for different positive integers, and confirm that they also end with 1.

The Collatz problem has been verified by computers for very large positive integers and though it has always held true, no mathematical proof has ever been found. It seems that the more we learn about this problem, the further away is its solution. This innocent looking puzzle is so complex that the legendary Hungarian-born mathematician Paul Erdös (1913-1996) commented: "Mathematics is not yet ready for such problems!" At this point the reader may ask himself: How can the $3n + 1$ problem be connected to the prime numbers? As will be shown in Chapter 9, even in mathematics there are pleasant surprises.

One of the oldest branches in pure mathematics is analytic number theory, in which the distribution of the primes among all natural numbers is a central concern. Euler once said: “Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate”. In Chapter 7, we will see whether Euler was right or wrong!